Common Stock Duration and Convexity

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Common stock duration and convexity are measures of the sensitivity of stock price to changes in discount rate. The discount rate is the risk-free interest rate plus a premium for risks applicable to holding common stock. A change in discount rate could come about from a change in the risk-free rate or a change in the equity risk premium. Duration and convexity are terms usually associated with bonds and the measurement of interest rate risk. Why would we be interested in the duration and convexity of common stock?

Imagine, if you will, that you are an investor with the opinion that the equity risk premium is too high and will revert to a number that is lower than the current level. You decide to buy the S&P 500 index (ticker SPY) because that is the index that most represents a well diversified common stock portfolio. You are betting that the discount rate decreases (because the equity risk premium decreases) and that the value of your investment rises because of that. This is similar to buying a BBB rated bond and having Moody's upgrade it to BB. The discount rate decreases (because the credit spread decreases) and results in a rise in bond price.

Your investment strategy has two problems. The first problem may be that you don't want to utilize the capital needed to take a large position in the S&P. The next problem is that you want to isolate the equity risk premium. The common stock discount rate includes both the risk-free rate and the equity risk premium. An unhedged position in the S&P 500 subjects you to interest rate risk as well as equity risk. The solution to both of these problems may be to short a risk-free bond (i.e. no default risk) and long the S&P 500. To properly hedge and isolate the equity risk premium we must short a risk-free bond that ***has the same duration*** as your long position in the S&P. Bond duration is easy to calculate because the bond cash flow, term and discount rate are known. Common stock duration is not so easy to calculate. Common stock cash flows are not contractual but are estimates, the term is unknown or infinite (cash flow in perpetuity) and the discount rate is unkown. So how do we calculate common stock duration?

Legend of Symbols

C_t	=	Cash flow per common share in period t
E_t	=	Earnings per common share at period t
P_t	=	Earnings payout ratio at period t
S_t	=	Stock price at period t
g	=	Earnings growth rate (annual)
$\hat{g}$	=	Earnings growth rate (continuous)
k	=	Equity discount rate (annual)
$\hat{k}$	=	Equity discount rate (continuous)
t	=	Time period
T	=	Total number of time periods

Stock Price Equation in Discrete Time

The assumption in discrete time will be that common stock receives cash flow annually. The variable t will therefore be year number and the variable T will be total number of years. Cash flow to common stock at the end of any year t will be...

$$C_t = E_t \times P_t \tag{1}$$

Stock price at time t is the discounted value of future cash flow. The equation for stock price at time zero is...

$$S_0 = \sum_{t=1}^{T} C_t \times (1+k)^{-t} \tag{2}$$

If we assume that both earnings growth rate and payout ratio are constant then equation (2) becomes...

$$S_0 = E_0 \times P \times \sum_{t=1}^{T} (1+g)^t \times (1+k)^{-t} \tag{3}$$

Stock Price Equation in Continous Time

When the move is made from discrete time to continuous time the annual growth rate and annual discount rate in discrete time must be converted to their continuous time equivalents. The following equations perform this conversion and defines a new variable theta (θ). Note that theta includes both discount rate and growth rate. For our purposes we will keep growth rate ($\hat{g}$) constant so the only change in theta will come from a change in discount rate ($\hat{k}$).

$$\hat{g} = \ln(1+g) \quad ; \quad \hat{k} = \ln(1+k) \quad ; \quad \theta = \hat{g} - \hat{k} \tag{4}$$

We now convert the stock price equation in discrete time (equation (3)) to its continuous time equivalent. The solution to the stock price integral is provided...

$$S_0 \quad = \quad E_0 \times P \times \int_0^T e^{\theta t} dt \quad = \quad E_0 \times P \times \left. \frac{e^{\theta t}}{\theta} \right]_0^T \quad = \quad E_0 \times P \times \frac{e^{\theta T} - 1}{\theta} \tag{5}$$

When the upper bound of integration (T) goes to infinity (i.e. a stock that pays dividends in perpetuity), then the stock price equation in continuous time where both the earnings growth rate and payout ratios are constant becomes...

$$S_0 = E_0 \times P \times \theta^{-1} \tag{6}$$

Using a Taylor Series Approximation, the change in stock price, ΔS, over some small interval, Δt, is given by the following partial differential equation (PDE)...

$$\Delta S \quad = \quad \frac{\delta S}{\delta t} \Delta t \quad + \quad \frac{\delta S}{\delta \theta} \Delta \theta \quad + \quad \frac{1}{2} \frac{\delta^2 S}{\delta \theta^2} (\Delta \theta)^2 \tag{7}$$

The first term in the PDE is $\frac{\delta S}{\delta t} \Delta t$, which is the first derivative of the price equation with respect to time (t). This partial derivative measures the change in price that occurs with the passage of time while holding discount rate constant. We are not interested in holding discount rate constant. We want to hold time constant and measure the change in price due to an instantaneous change in discount rate. For our purposes we can ignore this derivative.

The second term in the PDE is $\frac{\delta S}{\delta \theta} \Delta \theta$, which is the first derivative of the price equation with respect to theta. This partial derivative measures the first order change in price that occurs when discount rate changes while holding time (t) constant. We are interested in this deriviative because this is what we want to know ... how will stock price change with an instantaneous change in discount rate?

The third term in the PDE is $\frac{\delta^2 S}{\delta \theta^2} (\Delta \theta)^2$, which is the second derivative of the price equation with respect to theta and measures how stock duration changes as discount rate changes.

Stock Price Equation Derivatives

The first derivative of stock price equation (5) with respect to discount rate and its solution is...

$$\frac{\delta S}{\delta \theta} = E_0 \times P \times \int_0^T t e^{\theta t} dt = E_0 \times P \times \left. \frac{e^{\theta t}(\theta t - 1)}{\theta^2} \right]_0^T = E_0 \times P \times \frac{e^{\theta T}(\theta T - 1) - 1}{\theta^2} \tag{8}$$

The second derivative of stock price equation (5) with respect to discount rate and its solution is...

$$\frac{\delta^2 S}{\delta \theta^2} = E_0 \times P \times \int_0^T t^2 e^{\theta t} dt = E_0 \times P \times \left. \frac{e^{\theta t}(\theta t(\theta t - 2) + 2)}{\theta^3} \right]_0^T = E_0 \times P \times \frac{e^{\theta T}(e^{\theta T}(\theta T - 2) + 2) + 2)}{\theta^3} \tag{9}$$

Note that when T goes to infinity the first and second derivatives become...

$$\frac{\delta S}{\delta \theta} = E_0 \times P \times -\theta^{-2} \quad ...and... \quad \frac{\delta^2 S}{\delta \theta^2} = E_0 \times P \times -2\,\theta^{-3} \tag{10}$$

Common Stock Duration and Convexity

We now rewrite pertial differential equation (7) by dividing both sides by stock price such that new PDE describes the percentage change in stock price. The new PDE becomes...

$$\frac{\Delta S}{S} \quad = \quad \frac{1}{S}\frac{\delta S}{\delta t}\Delta t \quad + \quad \frac{1}{S}\frac{\delta S}{\delta \theta}\Delta \theta \quad + \quad \frac{1}{2}\frac{1}{S}\frac{\delta^2 S}{\delta \theta^2}(\Delta \theta)^2 \tag{11}$$

We will define $\frac{1}{S}\frac{\delta S}{\delta \theta}$ as common stock duration, which is equation (8) divided by equation (5). For a common stock paying dividends in perpetuity (T goes to infinity) the duration equation for that common stock becomes...

$$Duration \quad = \quad \frac{1}{S}\frac{\delta S}{\delta \theta} \quad = \quad \frac{E_0 \times P \times -\theta^{-2}}{E_0 \times P \times -\theta^{-1}} \quad = \quad \theta^{-1} \tag{12}$$

We will define $\frac{1}{S}\frac{\delta^2 S}{\delta \theta^2}$ as common stock convexity, which is equation (9) divided by equation (5). For a common stock paying dividends in perpetuity (T goes to infinity) the convexity equation for that common stock becomes...

$$Duration \quad = \quad \frac{1}{S}\frac{\delta^2 S}{\delta \theta^2} \quad = \quad \frac{E_0 \times P \times -2\,\theta^{-3}}{E_0 \times P \times -\theta^{-1}} \quad = \quad 2\,\theta^{-2} \tag{13}$$

After making these substitutions into equation (11) and then multiplying by stock price, equation (11) becomes...

$$\Delta S = S \times Duration \times \Delta \theta \quad + \quad \frac{1}{2} \times S \times Convexity \times (\Delta \theta)^2 \tag{14}$$

A Hypothetical Case

Problem: Calculate the duration and convexity for the S&P 500.

Inputs to the Duration and Convexity calculations:

k	=	11.50% (average total return)
g	=	5.00% (approximate GDP growth rate)
$\hat{k}$	=	0.1089
$\hat{g}$	=	0.0488
θ	=	-0.0601

Duration and Convexity calculation:

Duration	=	-16.65 annual periods
Convexity	=	554.37 annual periods